

Research Article

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A Proposed Model in which Solitons Exhibit Electron and Proton-like Behavior

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Abstract: A classical model is proposed in which two nonlinear Klein–Gordon fields interact via the electromagnetic field. Scaling is such that solitons in the two fields can be interpreted as electrons and protons, respectively. Even though the masses are very different, the magnitude of the charge of the electron-like soliton is the same as that of the proton-like soliton. Attraction and repulsion occur in the desired way through the interaction with the electromagnetic field.

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1 Introduction

Even from the time of de Broglie [14, 15] it has been frequently suggested that the notion of a point particle might be replaced by a small region in a field theory where the field is large. Regrettably, although it was easy and natural to find static solutions that had the right shape, the solutions always seemed to be unstable. From energy considerations Hobart [22] and Derrick [16] concluded that static, particle-like solutions are necessarily unstable for a wide class of possible equations. One approach to stability uses topological properties to achieve stability. The simplest example of that approach is the so-called sine-Gordon equation [37, 38].

In a second approach, however, nontopological solitons are considered. That approach, of interest here, uses a complex-valued dependent variable. The seeming inevitability of instability is avoided because the solution is not really static, but is spinning in the complex plane. The solution is stationary, nevertheless, in the sense that the modulus of the complex number remains constant in time. A simple example is the nonlinear Klein–Gordon equation (NKG)

$$\frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} - \nabla^2 \psi + W'(\psi \bar{\psi}) \psi = 0, \quad (1.1)$$

where $\psi(t, x)$ is a complex-valued, scalar field, and overbar indicates the complex conjugate. The spatial variable x is in \mathbb{R}^n with n typically 3, although $n = 1$ and $n = 2$ are sometimes of interest. The function $W: \mathbb{R} \rightarrow \mathbb{R}$ is used to introduce an appropriate nonlinearity, and $W'(\psi \bar{\psi})$ denotes the derivative of $W(\psi \bar{\psi})$ with respect to its argument $\psi \bar{\psi}$. We will take $W(0) = 0$, and with suitable scaling we can also take $W'(0) = 1$. Glasko et al. [21], Zastavenko [42], Rosen [36], and others did early work on the NKG equation using this approach. Extensive existence results have been obtained regarding localized, radially symmetric NKG solutions when certain conditions are imposed on the function W . When it is desired to investigate specific examples it is sometimes expedient to let W be piecewise linear. In that case W' is a step function, and various solutions can even be expressed explicitly [18, 27, 28, 36]. More recently, solutions with nonzero angular momentum have also been of considerable interest [4, 6].

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Localized solutions of this nature are often referred to as solitary waves, a term inherited from water wave theory [41]. A meaningful particle-like solution should also exhibit orbital stability, however, in which case the solution is then commonly referred to as a soliton. Orbital stability for the NKG equation has been considered by Shatah [40], Bellazzini et al. [5] and others.

An important step toward a physically relevant model is to couple the NKG equation to Maxwell's equations and thus to obtain the nonlinear Klein–Gordon–Maxwell equations (NKG-M), as was done by Rosen [35], Morris [31] and others. Existence results are available [7, 8, 25], some of these even in the case of nonzero angular momentum [9]. Despite the difficulties inherent in NKG-M, heartening stability results have been obtained by Long [25] and by Benci and Fortunato [10, 11], so that soliton solutions are known to exist at least in certain circumstances. The effect of external fields on soliton motion has also been investigated [2, 13, 26].

The nonlinear Schrödinger equation (NS), which is the nonrelativistic version of NKG, has also been of interest, especially in its relationship to the ideas of de Broglie and Bohm [1, 15]. Nontopological solitons have been considered in many contexts, some quantum-mechanical, some dealing with cosmological models or the study of the early universe. The literature on the subject is now very extensive, and the reader is referred to various review articles and books [23, 24, 29, 30, 39]. For references to some of the later work one may also see [3, 10, 11].

2 Model One

In the present paper we point out that, in addition to (1.1), it is useful to include a second NKG equation

$$\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi + M^2 W' \left(\frac{\phi \bar{\phi}}{M^2} \right) \phi = 0, \quad (2.1)$$

also coupled to the electromagnetic field, but scaled in such a way that solitons in the ϕ field can be thought of as protons whereas solitons in the ψ field are to be regarded as electrons. The constant M in (2.1) should be chosen (approximately 1836) so that the ratio of masses is correct; however, the scaling is such that, as desired, the magnitude of the charge turns out to be the same for the electron-like and proton-like solitons. We will calculate the force experienced when solitons interact at a distance through the electromagnetic field and show that like charges repel and opposite charges attract, as desired. Further, a criterion (related to the choice of W) will be determined for the interaction to be the right magnitude so as to agree with Coulomb's law.

The model proposed here could be referred to as a Double Nonlinear Klein–Gordon–Maxwell model (DNKGM). We prefer, however, to refer to it as Model One, both for simplicity and also to suggest that it is a first step (but only a first step!) toward a physically meaningful model.

As an alternate notation, let us now use a time coordinate $x^0 = ct$ and spatial coordinates x^1, x^2, x^3 and write

$$\psi_{,\alpha\beta} g^{\alpha\beta} + W'(\psi \bar{\psi}) \psi = 0, \quad \phi_{,\alpha\beta} g^{\alpha\beta} + M^2 W' \left(\frac{\phi \bar{\phi}}{M^2} \right) \phi = 0, \quad (2.2)$$

where $g^{\alpha\beta}$ is the metric tensor with $g^{00} = 1$, $g^{11} = g^{22} = g^{33} = -1$ and other entries zero. Indices after a comma designate partial derivatives, and repeated Greek indices are summed over 0, 1, 2, 3. The Lagrangian density for the electromagnetic field can be written

$$L = -\frac{1}{4} (A_{\alpha,\beta} - A_{\beta,\alpha})(A_{\kappa,\mu} - A_{\mu,\kappa}) g^{\alpha\kappa} g^{\beta\mu}, \quad (2.3)$$

where A_α is the four-dimensional vector potential. The Lagrangian densities for (2.2) are

$$\frac{1}{2} (\psi_{,\beta} \bar{\psi}_{,\kappa} g^{\beta\kappa} - W(\psi \bar{\psi})) \quad \text{and} \quad \frac{1}{2} (\phi_{,\beta} \bar{\phi}_{,\kappa} g^{\beta\kappa} - M^4 W(\phi \bar{\phi}/M^2)).$$

They need to be coupled to (2.3) in the usual gauge invariant way, with $\psi_{,\beta}$ replaced by $\psi_{,\beta} - iA_\beta\psi$ and $\phi_{,\beta}$ replaced by $\phi_{,\beta} - iA_\beta\phi$. Then the final Lagrangian density for Model One is

$$L = \frac{1}{2}(\psi_{,\beta} - iA_\beta\psi)(\overline{\psi}_{,\kappa} + iA_\kappa\overline{\psi})g^{\beta\kappa} - \frac{1}{2}W(\psi\overline{\psi}) + \frac{1}{2}(\phi_{,\beta} - iA_\beta\phi)(\overline{\phi}_{,\kappa} + iA_\kappa\overline{\phi})g^{\beta\kappa} - \frac{1}{2}M^4W\left(\frac{\phi\overline{\phi}}{M^2}\right) - \frac{1}{4}(A_{\alpha,\beta} - A_{\beta,\alpha})(A_{\kappa,\mu} - A_{\mu,\kappa})g^{\alpha\kappa}g^{\beta\mu}. \quad (2.4)$$

It is convenient to abbreviate

$$\psi_{;\beta} = \psi_{,\beta} - iA_\beta\psi, \quad \phi_{;\beta} = \phi_{,\beta} - iA_\beta\phi, \quad F_{\alpha\beta} = A_{\alpha,\beta} - A_{\beta,\alpha}, \quad F^{\alpha\beta} = F_{\kappa\mu}g^{\alpha\kappa}g^{\beta\mu},$$

so that (2.4) can be rewritten as

$$L = \frac{1}{2}\psi_{;\beta}\overline{\psi}_{;\kappa}g^{\beta\kappa} - \frac{1}{2}W(\psi\overline{\psi}) + \frac{1}{2}\phi_{;\beta}\overline{\phi}_{;\kappa}g^{\beta\kappa} - \frac{1}{2}M^4W\left(\frac{\phi\overline{\phi}}{M^2}\right) - \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}. \quad (2.5)$$

When the Euler operators Op_ψ , Op_ϕ , and Op_{A_α} are set to zero, the Euler equations turn out to be

$$\overline{Op}_\psi = -g^{\alpha\beta}\left(\frac{\partial}{\partial x^\alpha} - iA_\alpha\right)\left(\frac{\partial}{\partial x^\beta} - iA_\beta\right)\psi - W'(\psi\overline{\psi})\psi = 0, \quad (2.6)$$

$$\overline{Op}_\phi = -g^{\alpha\beta}\left(\frac{\partial}{\partial x^\alpha} - iA_\alpha\right)\left(\frac{\partial}{\partial x^\beta} - iA_\beta\right)\phi - M^2W'\left(\frac{\phi\overline{\phi}}{M^2}\right)\phi = 0, \quad (2.7)$$

$$Op_{A_\alpha} = \frac{\partial}{\partial x^\beta}F^{\alpha\beta} - \frac{i}{2}\overline{\psi}_{;\kappa}\psi g^{\alpha\kappa} + \frac{i}{2}\psi_{;\kappa}\overline{\psi}g^{\alpha\kappa} - \frac{i}{2}\overline{\phi}_{;\kappa}\phi g^{\alpha\kappa} + \frac{i}{2}\phi_{;\kappa}\overline{\phi}g^{\alpha\kappa} = 0. \quad (2.8)$$

The variational principle based on (2.5) is invariant under a gauge transformation; that is, ψ , ϕ , and A_β can be replaced by $\psi \exp(i\Phi)$, $\phi \exp(i\Phi)$, and $A_\beta + \Phi_{,\beta}$ respectively, where Φ is a real function of x^β . Then an equation in conservation form can be found in the standard way according to Noether's theorem [20, 32, 41] as

$$\begin{aligned} & \frac{i\Phi}{2}(\psi Op_\psi - \overline{\psi} Op_\psi + \phi Op_\phi - \overline{\phi} Op_\phi) + \Phi_{,\alpha} Op_{A_\alpha} \\ &= \frac{\partial}{\partial x^\mu} \left(\frac{i\Phi}{2}(-\psi\overline{\psi}_{;\alpha} + \overline{\psi}\psi_{;\alpha} - \phi\overline{\phi}_{;\alpha} + \overline{\phi}\phi_{;\alpha}) - F_{\alpha\beta}\Phi_{,\kappa}g^{\beta\kappa} \right) g^{\alpha\mu} = 0. \end{aligned} \quad (2.9)$$

The special case where Φ is taken as a constant gives the equation for conservation of charge, but it turns out that charge is conserved separately for ψ and ϕ as follows:

$$\begin{aligned} \frac{i}{2}(\psi Op_\psi - \overline{\psi} Op_\psi) &= \frac{i}{2}(-\psi\overline{\psi}_{;\kappa} + \overline{\psi}\psi_{;\kappa})_{,\beta}g^{\beta\kappa} = 0, \\ \frac{i}{2}(\phi Op_\phi - \overline{\phi} Op_\phi) &= \frac{i}{2}(-\phi\overline{\phi}_{;\kappa} + \overline{\phi}\phi_{;\kappa})_{,\beta}g^{\beta\kappa} = 0. \end{aligned} \quad (2.10)$$

Owing to translational invariance, a conservation equation for energy and momentum can also be found according to Noether's theorem as

$$\begin{aligned} & \frac{1}{2}\psi_{;\beta}Op_\psi + \frac{1}{2}\overline{\psi}_{;\beta}Op_\psi + \frac{1}{2}\phi_{;\beta}Op_\phi + \frac{1}{2}\overline{\phi}_{;\beta}Op_\phi + F_{\alpha\beta}Op_{A_\alpha} \\ &= \frac{\partial}{\partial x^\beta} \left(\frac{1}{2}\psi_{;\alpha}\overline{\psi}_{;\kappa}g^{\alpha\kappa} + \frac{1}{2}\phi_{;\alpha}\overline{\phi}_{;\kappa}g^{\alpha\kappa} - \frac{1}{2}W(\psi\overline{\psi}) - \frac{1}{2}M^4W\left(\frac{\phi\overline{\phi}}{M^2}\right) - \frac{1}{4}F_{\alpha\kappa}F^{\alpha\kappa} \right) \\ &+ \frac{\partial}{\partial x^\mu} \left(-\frac{1}{2}(\psi_{;\alpha}\overline{\psi}_{;\beta} + \overline{\psi}_{;\alpha}\psi_{;\beta} + \phi_{;\alpha}\overline{\phi}_{;\beta} + \overline{\phi}_{;\alpha}\phi_{;\beta})g^{\alpha\mu} + F^{\mu\nu}F_{\beta\nu} \right) = 0. \end{aligned} \quad (2.11)$$

Here $\beta = 0$ gives the energy conservation equation, and $\beta = 1, 2, 3$ give equations for conservation of momentum in the x^1, x^2, x^3 directions. The derivation of (2.11) can be done in the standard way, but the details are provided in Appendix A since the algebra is somewhat complicated.

3 Scaling to Relate to Real-World Phenomena

Let us first investigate solutions of the system (2.6)–(2.8) where ψ rotates in the complex plane and $\phi = 0$. Also we will take the vector field A_α as constant in time and will let its spatial components be zero, that is

$$\psi = U(x^1, x^2, x^3) \exp(i\omega x^0), \quad \phi = 0, \quad A_0 = A(x^1, x^2, x^3), \quad A_i = 0, \quad (3.1)$$

where $U: \mathbb{R}^3 \rightarrow \mathbb{R}$, $A: \mathbb{R}^3 \rightarrow \mathbb{R}$, $\omega \in \mathbb{R}$, and where Latin indices are to take the values 1, 2, 3. It follows that (2.7) is satisfied identically, and (2.6), (2.8) reduce to

$$U_{,ii} + (\omega - A)^2 U - W'(U^2)U = 0, \quad (3.2)$$

$$A_{,ii} = (-\omega + A)U^2, \quad (3.3)$$

respectively, with repeated Latin indices summed over 1, 2, 3.

First let us consider the linear partial differential equation that occurs in the limit of small U . Since we are taking $W'(0) = 1$, equation (3.2) becomes

$$U_{,ii} + ((\omega - A)^2 - 1)U = 0. \quad (3.4)$$

Although the present model is purely classical in nature, we expect some quantum-like behavior to occur, but in a different context from that of traditional quantum mechanics. Thus to relate the present model to real-world phenomena we want (3.4) to agree with the Klein–Gordon equation, so we choose the following scaling for the independent variables x^α :

$$ct = \frac{\hbar}{mc}x^0, \quad x = \frac{\hbar}{mc}x^1, \quad y = \frac{\hbar}{mc}x^2, \quad z = \frac{\hbar}{mc}x^3, \quad (3.5)$$

where t, x, y, z are time and space coordinates, m is the mass of the electron, c is the speed of light and \hbar is Planck's constant divided by 2π (all in customary units). Thus unit distance in our dimensionless coordinates corresponds to $\hbar/(mc)$, which is the Compton wavelength of the electron divided by 2π . That distance is about 3.86×10^{-13} meters, and defines a fundamental unit of length for quantum-mechanical phenomena that involve electrons. In problems of this nature it is usual to use “natural units” in which \hbar and the speed of light are scaled to unity (which can be done, for example, by defining new units of time and mass instead of seconds and kg). We choose to go further and nondimensionalize the equations completely by scaling elementary charge and electron mass also to unity (by defining new units of charge and distance instead of coulombs and meters). With such a scaling it turns out, for example, that the potential A is scaled in the Klein–Gordon equation in such a way that a nucleus with atomic number Z creates a potential well

$$A = -Z\alpha_{\text{fs}}/r, \quad (3.6)$$

where α_{fs} is the fine structure constant (approximately $1/137$) and

$$r^2 = x^i x^i = (x^1)^2 + (x^2)^2 + (x^3)^2.$$

4 Nonlinear Rotating Solutions

Now let us consider solutions of (3.2), (3.3) that have rather large U in a localized region (say near $r = 0$) but have U exponentially small when r is large. Such a solution is generally referred to as a solitary wave solution. Let us also assume that the solutions of interest have orbital stability properties that allow them to maintain their identity over time even in the presence of small perturbations. Such solutions are generally called particle-like solutions or solitons. When the conservation equation (2.10) is integrated (with volume element dV) over $r < R$ in \mathbb{R}^3 to give

$$\int \frac{i}{2}(-\psi\overline{\psi}_{;0} + \overline{\psi}\psi_{;0})_0 dV - \int \frac{i}{2}(-\psi\overline{\psi}_{;i} + \overline{\psi}\psi_{;i})_i dV = 0,$$

it is apparent from the divergence theorem that the second integral approaches zero for a soliton solution in the limit of large R . Then the charge, defined as

$$Q = \int_{\mathbb{R}^3} \frac{i}{2} (-\psi \bar{\psi}_{;0} + \bar{\psi} \psi_{;0}) dV,$$

must remain constant in time. For a soliton solution of the form (3.1) we have

$$Q = \int_{\mathbb{R}^3} (-\omega + A) U^2 dV. \quad (4.1)$$

In a similar way, the conservation equation (2.11) with $\beta = 0$ shows that the energy

$$\mathcal{E} = \int_{\mathbb{R}^3} \frac{1}{2} \left(\psi_{;0} \bar{\psi}_{;0} + \psi_{;i} \bar{\psi}_{;i} + \phi_{;0} \bar{\phi}_{;0} + \phi_{;i} \bar{\phi}_{;i} + W(\psi \bar{\psi}) + M^4 W\left(\frac{\phi \bar{\phi}}{M^2}\right) + F_{0i} F_{0i} + \frac{1}{2} F_{ij} F_{ij} \right) dV$$

remains constant in time, where it has been assumed that F_{ij} is $O(r^{-2})$ and that ψ and ϕ are exponentially small for large r . Then for a soliton solution of the form (3.1) the energy

$$\mathcal{E} = \int_{\mathbb{R}^3} \frac{1}{2} ((\omega - A)^2 U^2 + U_{,i} U_{,i} + W(U^2) + A_{,i} A_{,i}) dV \quad (4.2)$$

remains constant in time. For present purposes let us also assume that the soliton solutions of interest are spherically symmetric, in which case the substitutions $U = \mathcal{U}(r)$, $A = \mathcal{A}(r)$ reduce (3.2), (3.3) to ordinary differential equations.

For a soliton solution, (3.3) is effectively

$$A_{,ii} = 0$$

for sufficiently large r , where U is exponentially small. Then, for large r , A will be nearly proportional to $1/r$. Integrating (3.3) over volume and using the divergence theorem, we determine the constant of proportionality and find that in the limit of large r

$$A = -\frac{Q}{4\pi r},$$

where the charge Q is given by (4.1). For practical numerical investigations, it is sometimes useful to assume that A is small compared to U and thus to approximate (3.2) by

$$U_{,ii} + \omega^2 U - W'(U^2) U = 0. \quad (4.3)$$

For some simple forms of W , solutions of (4.3) can even be found analytically [36]. Then the full solution of the system (3.2), (3.3) can be sought by a numerical shooting method or by an iterative perturbation procedure.

5 Electron-Like Solitons

We want to interpret solitons of the form (3.1) as electrons, so we will refer to ψ as the electron field. When ω is positive in (3.1) we have arranged for Q to be negative in order to agree with the usual convention that the charge of an electron is negative. By comparison with (3.6) it is clear that, for an electron, A should approach α_{fs}/r for large r . Thus we need

$$Q = -4\pi\alpha_{fs}. \quad (5.1)$$

For a proposed initial choice of a function W , it may well be that condition (5.1) is not satisfied. We note, however, that if we have a solution for U in (4.3), the size of U can be changed (say multiplied by a factor β) by a suitable rescaling of W' with respect to its independent variable. Then, if A is small compared to ω ,

(3.3) shows that A will be approximately multiplied by the factor β^2 when the full numerical solution is obtained. In this way a suitable function W can generally be found in numerical work, and the size of U adjusted (perhaps after a few iterations) to give the desired value of Q for an electron-like soliton.

Suppose that an electron-like soliton solution of (3.2) and (3.3) is known with certain U , ω , and A . Such a solution rotates in the counterclockwise direction in the complex plane. Then a corresponding solution that rotates in the clockwise direction also occurs, with U , $-\omega$, $-A$. That solution has the same energy \mathcal{E} , but has one unit of elementary charge with opposite sign from that of the electron-like soliton. The revised solution with negative ω and positive Q is to be interpreted as a positron.

6 Difficulties

Two essential difficulties must be mentioned. First, for the NKGM system of equations (as well as for NKG) there is a whole family of solitary wave solutions with a range of values of Q . Within Model One it is not yet clear whether one preferred value of Q predominates in practice. Such a preferred value of Q would correspond to the elementary charge of an electron or proton. Morris [31] has offered a suggestion in this regard, but the problem urgently awaits further study.

Second, we are assuming here that, given an appropriate function W , the nonlinear rotating solution of interest will exhibit orbital stability. Then it can properly be termed a soliton. The rigorous stability proofs available at present apply in the limit of small coupling to the electromagnetic field. Further study will be needed to determine whether orbital stability is in fact achieved when the coupling is sufficient to correspond to the actual physical case.

7 Proton-Like Solitons

Now let us look for solutions of (2.6)–(2.8) of the form

$$\psi = 0, \quad \phi = \tilde{U}(x^1, x^2, x^3) \exp(-i\tilde{\omega}x^0), \quad A_0 = \tilde{A}(x^1, x^2, x^3), \quad A_i = 0, \quad (7.1)$$

where the real-valued functions \tilde{U} and \tilde{A} depend on x^i but not x^0 . Then \tilde{U} and \tilde{A} need to satisfy

$$\tilde{U}_{,ii} + (\tilde{\omega} + \tilde{A})^2 \tilde{U} - M^2 W' \left(\frac{\tilde{U}^2}{M^2} \right) \tilde{U} = 0, \quad \tilde{A}_{,ii} = (\tilde{\omega} + \tilde{A}) \tilde{U}^2. \quad (7.2)$$

If a solution $U = f(x^i)$ and $A = g(x^i)$ has been found for (3.2) and (3.3) with a certain value of ω , then

$$\tilde{U} = Mf(Mx^i), \quad \tilde{A} = -Mg(Mx^i) \quad (7.3)$$

gives a solution for (7.2) with $\tilde{\omega} = M\omega$. We have purposely introduced the minus sign in (7.1) so as to obtain clockwise rotation in the complex plane but still allow $\tilde{\omega}$ to be taken as positive. Calculating charge

$$\tilde{Q} = \int_{\mathbb{R}^3} \frac{i}{2} (-\phi \overline{\phi_{;0}} + \overline{\phi} \phi_{;0}) dV = \int_{\mathbb{R}^3} (\tilde{\omega} + \tilde{A}) \tilde{U}^2 dV$$

and energy

$$\tilde{\mathcal{E}} = \int_{\mathbb{R}^3} \frac{1}{2} \left((\tilde{\omega} + \tilde{A})^2 \tilde{U}^2 + \tilde{U}_{,i} \tilde{U}_{,i} + M^4 W \left(\frac{\tilde{U}^2}{M^2} \right) + \tilde{A}_{,i} \tilde{A}_{,i} \right) dV,$$

we find that the new solution has positive \tilde{Q} , with one unit of elementary charge, but its energy $\tilde{\mathcal{E}}$ is larger by a factor of M than that of the electron-like soliton.

We want to interpret a solution of the form (7.1) as a proton, so we will refer to ϕ as the proton field. Then we take M to be the appropriate value, approximately 1836, to give the desired mass ratio between the proton

and electron. Because of the scaling (7.3), the size, i.e., spatial extent, of the proton-like soliton is smaller by a factor of M than that of an electron-like soliton, even though the magnitude of the charge is the same. Thus, for the proton-like soliton, (3.6) with $Z = 1$ will be a good approximation except for very small values of r .

8 Momentum of a Moving Soliton

In order to see how a soliton is acted on by a force we need to find out how the momentum of a soliton depends on its velocity. To obtain the momentum conservation equation we replace β by j in (2.11). Let us consider an electron-like soliton (3.1) so that terms that involve ϕ can be set to zero. Then

$$\frac{\partial p_j}{\partial x^0} = \frac{\partial q_{ij}}{\partial x^i}, \quad (8.1)$$

where

$$\begin{aligned} p_j &= -\frac{1}{2}(\psi_{;0}\bar{\psi}_{;j} + \bar{\psi}_{;0}\psi_{;j}) + F_{0\kappa}F_{j\nu}g^{\kappa\nu}, \\ q_{ij} &= -\frac{1}{2}(\psi_{;i}\bar{\psi}_{;j} + \bar{\psi}_{;i}\psi_{;j}) + F_{i\kappa}F_{j\nu}g^{\kappa\nu} - \frac{1}{2}\left(\psi_{;\alpha}\bar{\psi}_{;\kappa}g^{\alpha\kappa} - W(\psi\bar{\psi}) - \frac{1}{2}F_{\alpha\kappa}F^{\alpha\kappa}\right)\delta_{ij} \end{aligned} \quad (8.2)$$

and δ_{ij} is the three-dimensional Kronecker delta, which is zero except for $\delta_{11} = \delta_{22} = \delta_{33} = 1$. To obtain the solution for an electron-like soliton moving with velocity v in the x^1 direction let us introduce a Lorentz boost; that is, we revise (3.1) as

$$\psi = U(\hat{x}^1, x^2, x^3) \exp(i\omega\hat{x}^0), \quad (8.3)$$

and

$$A_0 = \frac{A(\hat{x}^1, x^2, x^3)}{\sqrt{1-v^2}}, \quad A_1 = \frac{-vA(\hat{x}^1, x^2, x^3)}{\sqrt{1-v^2}}, \quad A_2 = A_3 = 0, \quad (8.4)$$

where \hat{x}^0 and \hat{x}^1 are given by the Lorentz transformation

$$\hat{x}^0 = \frac{x^0 - vx^1}{\sqrt{1-v^2}}, \quad \hat{x}^1 = \frac{x^1 - vx^0}{\sqrt{1-v^2}}.$$

After some calculation we find

$$p_1 = \left(\left(\frac{\partial U}{\partial \hat{x}^1} \right)^2 + (\omega - A)^2 U^2 + \left(\frac{\partial A}{\partial x^2} \right)^2 + \left(\frac{\partial A}{\partial x^3} \right)^2 \right) \frac{v}{1-v^2},$$

where we are still regarding $U = U(\hat{x}^1, x^2, x^3)$ and $A = A(\hat{x}^1, x^2, x^3)$ as expressed with the same arguments as in (8.3) and (8.4). Then the total momentum of the electron-like soliton is

$$P_1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_1 dx^1 dx^2 dx^3 = \sqrt{1-v^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_1 d\hat{x}^1 dx^2 dx^3$$

which can be written as

$$P_1 = \frac{v}{\sqrt{1-v^2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} ((U_{,1})^2 + (\omega - A)^2 U^2 + (A_{,2})^2 + (A_{,3})^2) dx^1 dx^2 dx^3$$

when $U = U(x^1, x^2, x^3)$ and $A = A(x^1, x^2, x^3)$ are regarded as expressed in terms of the original arguments. More concisely,

$$P_1 = \frac{v}{\sqrt{1-v^2}} \int ((U_{,1})^2 + (\omega - A)^2 U^2 + (A_{,2})^2 + (A_{,3})^2) dV,$$

where integrals that involve the volume element dV will be over \mathbb{R}^3 . In the situation of interest, U and A are spherically symmetric, i.e., $U = \mathcal{U}(r)$, $A = \mathcal{A}(r)$, where $r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$, so by symmetry

$$\begin{aligned} \int (U_{,1})^2 dV &= \int (U_{,2})^2 dV = \int (U_{,3})^2 dV = \frac{1}{3} \int U_{,i} U_{,i} dV, \\ \int (A_{,1})^2 dV &= \int (A_{,2})^2 dV = \int (A_{,3})^2 dV = \frac{1}{3} \int A_{,i} A_{,i} dV. \end{aligned}$$

Then

$$P_1 = \frac{v}{\sqrt{1-v^2}} \left(\frac{1}{3} I_1 + I_4 + \frac{2}{3} I_2 \right), \quad (8.5)$$

where

$$I_1 = \int U_{,i} U_{,i} dV = \int_0^\infty \mathcal{U}'(r)^2 4\pi r^2 dr, \quad (8.6)$$

$$I_2 = \int A_{,i} A_{,i} dV = \int_0^\infty \mathcal{A}'(r)^2 4\pi r^2 dr, \quad (8.7)$$

$$I_3 = \int W(U^2) dV = \int_0^\infty W(\mathcal{U}(r)^2) 4\pi r^2 dr, \quad (8.8)$$

$$I_4 = \int (\omega - A)^2 U^2 dV = \int_0^\infty (\omega - \mathcal{A}(r))^2 \mathcal{U}(r)^2 4\pi r^2 dr, \quad (8.9)$$

(with I_3 included to be used later). For low velocities, where $\sqrt{1-v^2}$ can be approximated by unity, we see that momentum is essentially mass times velocity: $P_1 = M_{\text{eff}} v$, and the effective mass turns out to be

$$M_{\text{eff}} = \frac{1}{3} I_1 + I_4 + \frac{2}{3} I_2. \quad (8.10)$$

The expression (8.10) for M_{eff} allows for the possibility that A is substantial, but if A indeed turns out to be negligibly small for the soliton solutions of interest, then (8.10) reduces to the expression used by Long and Stuart [26].

As desired for a proton-like soliton, M_{eff} is larger by a factor of M compared to M_{eff} for an electron-like soliton. That result can be seen from the above integrals since U , A , and ω are each larger by a factor of M , and each derivative introduces a further factor of M , but scaling of the independent variable r reduces the size of each volume integral by a factor of M^3 .

9 Interaction via the Electromagnetic Field

It turns out that electron-like and proton-like solitons interact through the electromagnetic field to give attraction and repulsion. Long and Stuart [26] have examined in detail the action of an external field on a soliton in terms of the Lorentz force law. Here we will use a less rigorous but much simpler approach since for present purposes we want merely to establish the relationship to Coulomb's law.

Let us consider the force on an electron-like soliton at rest at the origin when it is acted on by a small external potential $A^{(\text{ext})}$, and let us assume that the stability properties of the soliton allow it to maintain its identity even when it is perturbed by the external potential. Then (3.1) is changed so that

$$A_0 = A(x^1, x^2, x^3) + A^{(\text{ext})}, \quad A_i = 0. \quad (9.1)$$

Here we are assuming, when $A^{(\text{ext})}$ is sufficiently small, that $A(x^1, x^2, x^3)$ is not appreciably changed and that $A_i = 0$ is still a good approximation. From (8.1) and (8.2) we see that the force acting on the electron-like soliton is

$$\begin{aligned} \int \frac{\partial q_{ij}}{\partial x^i} dV &= \int \frac{\partial}{\partial x^i} \frac{1}{2} \left(-\psi_{,i} \bar{\psi}_{,j} - \bar{\psi}_{,i} \psi_{,j} - (\psi_{,a} \bar{\psi}_{,k} g^{ak} - W(\psi \bar{\psi})) \delta_{ij} \right) dV \\ &\quad + \int \frac{\partial}{\partial x^i} \left(F_{ik} F_{j\nu} g^{k\nu} + \frac{1}{4} F_{a\kappa} F^{a\kappa} \delta_{ij} \right) dV. \end{aligned} \quad (9.2)$$

Let us take the volume integrals for $r < R$ where R is large. Then the first integral on the right-hand side of (9.2) is negligible because it can be rewritten by means of the divergence theorem, and ψ is exponentially

small for large r . Using (9.1), we find

$$F_{00} = 0, \quad F_{0i} = -F_{i0} = A_{0,i}, \quad F_{ij} = 0,$$

so that (9.2) reduces to

$$\int \frac{\partial q_{ij}}{\partial x^i} dV = \int \frac{\partial}{\partial x^i} \left(A_{0,i} A_{0,j} - \frac{1}{2} A_{0,k} A_{0,k} \delta_{ij} \right) dV = \int A_{0,ii} A_{0,j} dV.$$

In situations of interest, let us approximate the external potential by a linear expression $A^{(\text{ext})} = a + b_j x^j$ where a and b_j are constant. Then

$$\int \frac{\partial q_{ij}}{\partial x^i} dV = \int A_{,ii} A_{,j} dV + \int b_j A_{,ii} dV. \quad (9.3)$$

The first integral on the right-hand side of (9.3) is zero because A is spherically symmetric. The constant b_j can be taken out of the second integral. Then, using (3.3) and (4.1), we see that (8.1) for $j = 1$ reduces to just

$$M_{\text{eff}} \frac{dv}{dx^0} = \int \frac{\partial q_{i1}}{\partial x^i} dV = b_1 Q.$$

Now suppose that the electron-like soliton at the origin is acted on by another electron-like soliton at position $x^1 = r$ on the x^1 axis, where r is a large positive number. We expect that $A^{(\text{ext})} = a + b_j x^j$ with $a = \alpha_{\text{fs}} r^{-1}$, $b_1 = \alpha_{\text{fs}} r^{-2}$, $b_2 = b_3 = 0$ will be an adequate approximation, and we find

$$M_{\text{eff}} \frac{dv}{dx^0} = \frac{\alpha_{\text{fs}} Q}{r^2},$$

where dv/dx^0 is the acceleration in the x^1 direction of the electron-like soliton at the origin. Checking the signs (and remembering that Q is negative), we see that the two electron-like solitons repel. Similarly one can show that two proton-like solitons repel, but an electron-like and a proton-like soliton attract each other. It is interesting that early attempts to study interactions [17, 34], using metastable particles in a real-valued NKG equation, gave *attraction* of like charges via the real-valued NKG field, whereas, for Model One, interaction via the electromagnetic A_α field automatically gives the desired *repulsion* of like charges.

The scaling (3.5) was originally set up as appropriate to the Klein–Gordon equation but without specific reference to Coulomb's law. In terms of that scaling, Coulomb's law for two electrons at distance r turns out to be simply

$$\frac{dv}{dx^0} = -\frac{\alpha_{\text{fs}}}{r^2}.$$

Thus we find that a further condition is needed in Model One so that Coulomb's law will hold: M_{eff} should turn out to be numerically very close to the absolute value of Q for the function W that is to be used.

10 Derivation of a Relation Between Integrals

When the variational principle obtained from the Lagrangian density (2.5) is reduced using (3.1) it becomes

$$J = \int \left(\frac{1}{2} (\omega - A)^2 U^2 - \frac{1}{2} W(U^2) - \frac{1}{2} U_{,i} U_{,i} + \frac{1}{2} A_{,i} A_{,i} \right) dV. \quad (10.1)$$

The Euler equations (3.2) and (3.3) are obtained by variation with respect to U and A , respectively. As before, U , A and ω will be assumed to give a localized solution of the Euler equations. Then a useful relation between integrals can be obtained in a standard way as follows: Given U and A let us define related functions $\hat{U}(x^1, x^2, x^3) = U(\lambda x^1, \lambda x^2, \lambda x^3)$ and $\hat{A}(x^1, x^2, x^3) = A(\lambda x^1, \lambda x^2, \lambda x^3)$ that also depend on a real parameter λ . If instead of U , A we substitute the new functions \hat{U} , \hat{A} in (10.1) we find that J also depends on the parameter λ , so that

$$J(\lambda) = \int \left(\frac{1}{2} (\omega - \hat{A})^2 \hat{U}^2 - \frac{1}{2} W(\hat{U}^2) - \frac{1}{2} \hat{U}_{,i} \hat{U}_{,i} + \frac{1}{2} \hat{A}_{,i} \hat{A}_{,i} \right) dV.$$

Changing variables in the integral in the following manner,

$$\iiint f(\lambda x^1, \lambda x^2, \lambda x^3) dx^1 dx^2 dx^3 = \lambda^{-3} \iiint f(\zeta^1, \zeta^2, \zeta^3) d\zeta^1 d\zeta^2 d\zeta^3,$$

we find that $J(\lambda)$ can be rewritten as

$$\lambda^{-3} \int \left(\frac{1}{2} (\omega - A)^2 U^2 - \frac{1}{2} W(U^2) \right) dV - \lambda^{-1} \int \left(\frac{1}{2} U_{,i} U_{,i} - \frac{1}{2} A_{,i} A_{,i} \right) dV,$$

where the derivatives of \hat{U} and \hat{A} account for the extra factor of λ^2 in the second integral. Since J is stationary with respect to variations δU and δA , we can then set $2J'(1) = 0$ to obtain

$$I_1 - I_2 + 3I_3 - 3I_4 = 0, \quad (10.2)$$

which is written in terms of the integrals (8.6)–(8.9). Special cases of (10.2) relevant to the NKG equation are well known in the literature [3, 12, 19], where they are often associated with the names of Derrick [16] and Pohožaev. Rosen [33] refers to such results as pseudovirial theorems. The more general case (10.2), which includes coupling to the A_α field and is needed in the NKGm context, is less well known but has been occasionally mentioned [25]. It is instructive (although somewhat more complicated) to see that (10.2) holds as a consequence of Lorentz invariance. The details are in Appendix B.

11 Energy of a Moving Soliton

The energy of a stationary soliton (4.2) can be rewritten as

$$\mathcal{E} = \frac{1}{2} (I_4 + I_1 + I_3 + I_2)$$

in terms of the integrals (8.6)–(8.9). Performing a calculation similar to that in Section 8 we find that the energy of a soliton moving with velocity v is

$$\mathcal{E} = \frac{I_4 + I_1 + I_3 + I_2}{2\sqrt{1-v^2}} + \frac{(3I_4 - I_1 - 3I_3 + I_2)v^2}{6\sqrt{1-v^2}},$$

which can be simplified using (10.2) to

$$\mathcal{E} = \frac{1}{\sqrt{1-v^2}} \left(\frac{1}{3} I_1 + I_4 + \frac{2}{3} I_2 \right). \quad (11.1)$$

It is apparent from (8.5) and (11.1) that, for a moving soliton, energy and momentum transform as the components of a Lorentz four-vector. That result was shown by Dudnikova et al. [19] and Badiale et al. [3] in the NKG case, but here we see that it applies more generally, even in the NKGm case where there is coupling to the electromagnetic field. Thus, for a soliton, we find in particular that the rest energy, (11.1) with $v = 0$, is equal to M_{eff} (a result that is more familiar in customary units as $E = mc^2$).

12 Summary and Conclusions

Model One (2.4) presents an appealing picture in which electron-like and proton-like solitons have appropriate charge and mass so that they can attract and repel in the desired manner. As mentioned in Section 6, however, a chief concern is that there is typically a whole family of solitons with various values of charge Q , so it is not immediately clear that Model One results in a single, definite value for the elementary charge. Further study of stability is also eagerly awaited.

From the time of Schrödinger and Dirac the deficiencies of the Klein–Gordon equation have been well known, so a revised model (Model Two) is being considered in which the two NKG equations will be replaced by nonlinear Dirac equations. Even if such a model is fairly straightforward to define, it will obviously be much more difficult to study in any detail. Although Model One has known deficiencies, it appears, owing to its relative simplicity, that interesting and useful insights can be obtained by its further study.

A Energy-Momentum Equation

To obtain (2.11) we first work out the energy-momentum conservation equation that results directly from translational invariance in time and space. Using (2.6)–(2.8), we write out the expression

$$\frac{1}{2}\psi_{,\beta}Op\psi + \frac{1}{2}\overline{\psi}_{,\beta}\overline{Op\psi} + \frac{1}{2}\phi_{,\beta}Op\phi + \frac{1}{2}\overline{\phi}_{,\beta}\overline{Op\phi} + A_{\alpha,\beta}OpA_{\alpha} \quad (\text{A.1})$$

as

$$\begin{aligned} & \frac{1}{2}\psi_{,\beta}g^{\alpha\kappa}(-\overline{\psi}_{,\alpha\kappa} - iA_{\alpha,\kappa}\overline{\psi} - 2iA_{\alpha}\overline{\psi}_{,\kappa} + A_{\alpha}A_{\kappa}\overline{\psi}) - \frac{1}{2}\psi_{,\beta}W'(\psi\overline{\psi})\overline{\psi} \\ & + \frac{1}{2}\overline{\psi}_{,\beta}g^{\alpha\kappa}(-\psi_{,\alpha\kappa} + iA_{\alpha,\kappa}\psi + 2iA_{\alpha}\psi_{,\kappa} + A_{\alpha}A_{\kappa}\psi) - \frac{1}{2}\overline{\psi}_{,\beta}W'(\psi\overline{\psi})\psi \\ & + \frac{1}{2}\phi_{,\beta}g^{\alpha\kappa}(-\overline{\phi}_{,\alpha\kappa} - iA_{\alpha,\kappa}\overline{\phi} - 2iA_{\alpha}\overline{\phi}_{,\kappa} + A_{\alpha}A_{\kappa}\overline{\phi}) - \frac{1}{2}\phi_{,\beta}M^2W'\left(\frac{\phi\overline{\phi}}{M^2}\right)\overline{\phi} \\ & + \frac{1}{2}\overline{\phi}_{,\beta}g^{\alpha\kappa}(-\phi_{,\alpha\kappa} + iA_{\alpha,\kappa}\phi + 2iA_{\alpha}\phi_{,\kappa} + A_{\alpha}A_{\kappa}\phi) - \frac{1}{2}\overline{\phi}_{,\beta}M^2W'\left(\frac{\phi\overline{\phi}}{M^2}\right)\phi \\ & + A_{\alpha,\beta}\left((A_{\kappa,\lambda\mu} - A_{\lambda,\kappa\mu})g^{\alpha\kappa}g^{\lambda\mu} - \frac{i}{2}(\overline{\psi}_{,\kappa} + iA_{\kappa}\overline{\psi})\psi g^{\alpha\kappa} + \frac{i}{2}(\psi_{,\kappa} - iA_{\kappa}\psi)\overline{\psi}g^{\alpha\kappa}\right. \\ & \quad \left. - \frac{i}{2}(\overline{\phi}_{,\kappa} + iA_{\kappa}\overline{\phi})\phi g^{\alpha\kappa} + \frac{i}{2}(\phi_{,\kappa} - iA_{\kappa}\phi)\overline{\phi}g^{\alpha\kappa}\right), \end{aligned}$$

where we have freely used $g^{\alpha\kappa} = g^{\kappa\alpha}$. Grouping terms differently, we have

$$\begin{aligned} & -\frac{1}{2}g^{\alpha\kappa}\psi_{,\beta}\overline{\psi}_{,\alpha\kappa} - \frac{1}{2}g^{\alpha\kappa}\overline{\psi}_{,\beta}\psi_{,\alpha\kappa} - \frac{1}{2}W'(\psi\overline{\psi})(\psi_{,\beta}\overline{\psi} + \psi\overline{\psi}_{,\beta}) \\ & + \frac{i}{2}g^{\alpha\kappa}A_{\alpha,\kappa}(-\psi_{,\beta}\overline{\psi} + \overline{\psi}_{,\beta}\psi) + ig^{\alpha\kappa}A_{\alpha}(-\psi_{,\beta}\overline{\psi}_{,\kappa} + \overline{\psi}_{,\beta}\psi_{,\kappa}) - \frac{i}{2}A_{\alpha,\beta}(\overline{\psi}_{,\kappa}\psi - \psi_{,\kappa}\overline{\psi})g^{\alpha\kappa} \\ & + \frac{1}{2}g^{\alpha\kappa}A_{\alpha}A_{\kappa}(\psi_{,\beta}\overline{\psi} + \overline{\psi}_{,\beta}\psi) + g^{\alpha\kappa}A_{\alpha,\beta}A_{\kappa}\overline{\psi}\psi \\ & - \frac{1}{2}g^{\alpha\kappa}\phi_{,\beta}\overline{\phi}_{,\alpha\kappa} - \frac{1}{2}g^{\alpha\kappa}\overline{\phi}_{,\beta}\phi_{,\alpha\kappa} - \frac{1}{2}M^2W'\left(\frac{\phi\overline{\phi}}{M^2}\right)(\phi_{,\beta}\overline{\phi} + \phi\overline{\phi}_{,\beta}) \\ & + \frac{i}{2}g^{\alpha\kappa}A_{\alpha,\kappa}(-\phi_{,\beta}\overline{\phi} + \overline{\phi}_{,\beta}\phi) + ig^{\alpha\kappa}A_{\alpha}(-\phi_{,\beta}\overline{\phi}_{,\kappa} + \overline{\phi}_{,\beta}\phi_{,\kappa}) \\ & - \frac{i}{2}g^{\alpha\kappa}A_{\alpha,\beta}(\overline{\phi}_{,\kappa}\phi - \phi_{,\kappa}\overline{\phi}) + \frac{1}{2}g^{\alpha\kappa}A_{\alpha}A_{\kappa}(\phi_{,\beta}\overline{\phi} + \overline{\phi}_{,\beta}\phi) + g^{\alpha\kappa}A_{\alpha,\beta}A_{\kappa}\overline{\phi}\phi \\ & + A_{\alpha,\beta}A_{\kappa,\mu\lambda}g^{\alpha\kappa}g^{\mu\lambda} - A_{\alpha,\beta}A_{\mu,\kappa\lambda}g^{\alpha\kappa}g^{\mu\lambda}. \quad (\text{A.2}) \end{aligned}$$

To obtain the desired equation in conservation form we need to rewrite the various terms as derivatives. The first two terms of (A.2) can be rewritten as

$$\begin{aligned} & -\left(\frac{1}{2}g^{\alpha\kappa}\psi_{,\beta}\overline{\psi}_{,\alpha}\right)_{,\kappa} + \frac{1}{2}g^{\alpha\kappa}\psi_{,\beta\kappa}\overline{\psi}_{,\alpha} - \left(\frac{1}{2}g^{\alpha\kappa}\overline{\psi}_{,\beta}\psi_{,\alpha}\right)_{,\kappa} + \frac{1}{2}g^{\alpha\kappa}\overline{\psi}_{,\beta\kappa}\psi_{,\alpha} \\ & = -\frac{1}{2}g^{\alpha\kappa}(\psi_{,\beta}\overline{\psi}_{,\alpha} + \overline{\psi}_{,\beta}\psi_{,\alpha})_{,\kappa} + \left(\frac{1}{2}g^{\alpha\kappa}\psi_{,\kappa}\overline{\psi}_{,\alpha}\right)_{,\beta}, \end{aligned}$$

where we have used $g^{\alpha\kappa}\psi_{,\kappa}\overline{\psi}_{,\alpha} = g^{\alpha\kappa}\psi_{,\alpha}\overline{\psi}_{,\kappa}$ since $g^{\alpha\kappa} = g^{\kappa\alpha}$. The third term in (A.2) can be rewritten as $-(1/2)W(\psi\overline{\psi})_{,\beta}$. The fourth term in (A.2) can be rewritten as

$$\begin{aligned} & \frac{\partial}{\partial x^{\kappa}}\left(\frac{i}{2}g^{\alpha\kappa}A_{\alpha}(-\psi_{,\beta}\overline{\psi} + \overline{\psi}_{,\beta}\psi)\right) - \frac{i}{2}g^{\alpha\kappa}A_{\alpha}\frac{\partial}{\partial x^{\kappa}}(-\psi_{,\beta}\overline{\psi} + \overline{\psi}_{,\beta}\psi) \\ & = \frac{\partial}{\partial x^{\kappa}}\left(\frac{i}{2}g^{\alpha\kappa}A_{\alpha}(-\psi_{,\beta}\overline{\psi} + \overline{\psi}_{,\beta}\psi)\right) - \frac{i}{2}g^{\alpha\kappa}A_{\alpha}(-\psi_{,\beta\kappa}\overline{\psi} - \psi_{,\beta}\overline{\psi}_{,\kappa} + \overline{\psi}_{,\beta\kappa}\psi + \psi_{,\beta}\overline{\psi}_{,\kappa}). \end{aligned}$$

When the sixth term is treated similarly, the fourth, fifth, and sixth terms reduce to the desired form after several terms cancel. The seventh and eighth terms can be rewritten as $(g^{\alpha\kappa}A_{\alpha}A_{\kappa}\psi\overline{\psi}/2)_{,\beta}$. The corresponding

terms in (A.2) that involve ϕ can be manipulated in a similar way. Finally, the last two terms in (A.2) can be rewritten as

$$\begin{aligned} & (A_{\alpha,\beta}A_{\kappa,\mu}g^{\alpha\kappa}g^{\mu\lambda})_{,\lambda} - A_{\alpha,\beta\lambda}A_{\kappa,\mu}g^{\alpha\kappa}g^{\mu\lambda} - (A_{\alpha,\beta}A_{\mu,\kappa}g^{\alpha\kappa}g^{\mu\lambda})_{,\lambda} + A_{\alpha,\lambda\beta}A_{\mu,\kappa}g^{\alpha\kappa}g^{\mu\lambda} \\ &= (A_{\alpha,\beta}A_{\kappa,\mu}g^{\alpha\kappa}g^{\mu\lambda})_{,\lambda} - \frac{\partial}{\partial x^\beta} \left(\frac{1}{2}A_{\alpha,\lambda}A_{\kappa,\mu}g^{\alpha\kappa}g^{\mu\lambda} \right) - (A_{\alpha,\beta}A_{\mu,\kappa}g^{\alpha\kappa}g^{\mu\lambda})_{,\lambda} + \frac{\partial}{\partial x^\beta} \left(\frac{1}{2}A_{\alpha,\lambda}A_{\mu,\kappa}g^{\alpha\kappa}g^{\mu\lambda} \right). \end{aligned}$$

Collecting the various terms and setting the expression for (A.1) to zero, we obtain an equation in conservation form:

$$\begin{aligned} & \frac{\partial}{\partial x^\beta} \left(\frac{1}{2}g^{\alpha\kappa}\psi_{;\kappa}\bar{\psi}_{;\alpha} + \frac{1}{2}g^{\alpha\kappa}\phi_{;\kappa}\bar{\phi}_{;\alpha} - \frac{1}{2}W(\psi\bar{\psi}) - \frac{1}{2}M^4W\left(\frac{\phi\bar{\phi}}{M^2}\right) - \frac{1}{2}A_{\alpha,\lambda}F_{\kappa\mu}g^{\alpha\kappa}g^{\mu\lambda} \right) \\ &+ \frac{\partial}{\partial x^\mu} \left(-\frac{1}{2}g^{\alpha\mu}(\psi_{;\beta}\bar{\psi}_{;\alpha} + \bar{\psi}_{;\beta}\psi_{;\alpha}) - \frac{1}{2}g^{\alpha\mu}(\phi_{;\beta}\bar{\phi}_{;\alpha} + \bar{\phi}_{;\beta}\phi_{;\alpha}) + A_{\alpha,\beta}F_{\kappa\lambda}g^{\alpha\kappa}g^{\mu\lambda} \right) = 0. \end{aligned} \quad (\text{A.3})$$

Although (A.3) gives a conservation equation for energy and momentum, it is desirable to modify the result to incorporate gauge invariance. First we obtain a special case of (2.9) by substituting A_β formally for Φ to get

$$\frac{\partial}{\partial x^\mu} \left(\frac{A_\beta}{2i} (\psi\bar{\psi}_{;\alpha} - \bar{\psi}\psi_{;\alpha} + \phi\bar{\phi}_{;\alpha} - \bar{\phi}\phi_{;\alpha}) - F_{\alpha\lambda}A_{\beta,\kappa}g^{\lambda\kappa} \right) g^{\alpha\mu} = 0. \quad (\text{A.4})$$

Alternatively we can obtain (A.4) directly by setting the expression

$$\frac{A_\beta}{2i} (-\psi Op_\psi + \bar{\psi} Op_\psi - \phi Op_\phi + \bar{\phi} Op_\phi) + A_{\beta,\alpha} Op_{A_\alpha}$$

to zero. Finally, we subtract (A.4) from (A.3) to obtain the energy-momentum conservation equation in the desired form (2.11).

B Derivation Using Lorentz Invariance

Let us consider the reduced form of (2.5) where ϕ dependence is omitted. According to Noether's theorem, it is possible to find a conservation equation that corresponds to Lorentz invariance in the x^0, x^1 -plane. We set the expression

$$\frac{1}{2}(\psi_{,0}x^1 + \psi_{,1}x^0)Op_\psi + \frac{1}{2}(\bar{\psi}_{,0}x^1 + \bar{\psi}_{,1}x^0)\overline{Op}_\psi + (A_{\alpha,0}x^1 + A_{\alpha,1}x^0)Op_{A_\alpha} + Op_{A_0}A_1 + Op_{A_1}A_0$$

to zero and rewrite suitable terms in integrated form to obtain the conservation equation

$$\begin{aligned} & \frac{\partial}{\partial x^0} \left(\frac{1}{2}\psi_{;\alpha}\bar{\psi}_{;\beta}g^{\alpha\beta} - \frac{1}{2}W(\psi\bar{\psi}) \right) x^1 + \frac{\partial}{\partial x^1} \left(\frac{1}{2}\psi_{;\alpha}\bar{\psi}_{;\beta}g^{\alpha\beta} - \frac{1}{2}W(\psi\bar{\psi}) \right) x^0 \\ & - \frac{1}{2} \frac{\partial}{\partial x^\alpha} (\psi_{,0}\bar{\psi}_{;\beta}x^1 + \bar{\psi}_{,0}\psi_{;\beta}x^1 + \psi_{,1}\bar{\psi}_{;\beta}x^0 + \bar{\psi}_{,1}\psi_{;\beta}x^0) g^{\alpha\beta} \\ & + (A_{\kappa,\mu}A_{\alpha,0}g^{\mu\beta}g^{\alpha\kappa}x^1)_{,\beta} - \left(\frac{1}{2}A_{\kappa,\mu}A_{\alpha,\beta}g^{\mu\beta}g^{\alpha\kappa}x^1 \right)_{,0} \\ & - (A_{\mu,\beta}A_{\alpha,0}g^{\mu\beta}g^{\alpha\kappa}x^1)_{,\kappa} + \left(\frac{1}{2}A_{\mu,\beta}A_{\alpha,\kappa}g^{\mu\beta}g^{\alpha\kappa}x^1 \right)_{,0} \\ & + (A_{\kappa,\mu}A_{\alpha,1}g^{\mu\beta}g^{\alpha\kappa}x^0)_{,\beta} - \left(\frac{1}{2}A_{\kappa,\mu}A_{\alpha,\beta}g^{\mu\beta}g^{\alpha\kappa}x^0 \right)_{,1} \\ & - (A_{\mu,\beta}A_{\alpha,1}g^{\mu\beta}g^{\alpha\kappa}x^0)_{,\kappa} + \left(\frac{1}{2}A_{\mu,\beta}A_{\alpha,\kappa}g^{\mu\beta}g^{\alpha\kappa}x^0 \right)_{,1} \\ & + (A_{0,\mu}A_{\alpha,1}g^{\mu\beta})_{,\beta} - (A_{1,\mu}A_{\alpha,0}g^{\mu\beta})_{,\beta} - (A_{\mu,\beta}A_{\alpha,1}g^{\mu\beta})_{,0} + (A_{\mu,\beta}A_{\alpha,0}g^{\mu\beta})_{,1} = 0. \end{aligned} \quad (\text{B.1})$$

For simplicity let us write x^0, x^1, x^2, x^3 as t, x, y, z , respectively, and denote partial derivatives by subscripts. Substituting

$$u = U \exp(i\omega t), \quad A_0 = A, \quad A_i = 0,$$

where $U = U(x, y, z)$, $A = A(x, y, z)$ do not depend on t , we find that (B.1) reduces to

$$\begin{aligned} & \frac{\partial}{\partial t}(-U^2\omega^2 - U_x^2 - U_y^2 - U_z^2 - W(U^2) + U^2A^2 + A_x^2 + A_y^2 + A_z^2)\frac{x}{2} \\ & + \frac{\partial}{\partial x}(U_x^2 - U_y^2 - U_z^2 - W(U^2) + (\omega - A)^2U^2 - A_x^2 + A_y^2 + A_z^2)\frac{t}{2} \\ & + \frac{\partial}{\partial y}(U_yU_x - A_yA_x)t + \frac{\partial}{\partial z}(U_zU_x - A_zA_x)t = 0. \end{aligned} \quad (\text{B.2})$$

The t derivative term in (B.2) is zero since the expression acted on is independent of t . Thus after omitting the t derivative term we can divide (B.2) by t (or set $t = 1$) to get

$$\begin{aligned} & \frac{\partial}{\partial x}\frac{1}{2}(U_x^2 - U_y^2 - U_z^2 - W(U^2) + (\omega - A)^2U^2 - A_x^2 + A_y^2 + A_z^2) \\ & + \frac{\partial}{\partial y}(U_yU_x - A_yA_x) + \frac{\partial}{\partial z}(U_zU_x - A_zA_x) = 0. \end{aligned}$$

Now let us integrate over x from $-\infty$ to \hat{x} , over y from $-\infty$ to ∞ , and over z from $-\infty$ to ∞ to get

$$\begin{aligned} & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\hat{x}} \frac{\partial}{\partial x}\frac{1}{2}(U_x^2 - U_y^2 - U_z^2 - W(U^2) + (\omega - A)^2U^2 - A_x^2 + A_y^2 + A_z^2)dx dy dz \\ & + \int_{-\infty}^{\infty} \int_{-\infty}^{\hat{x}} \int_{-\infty}^{\infty} \frac{\partial}{\partial y}(U_yU_x - A_yA_x)dy dx dz + \int_{-\infty}^{\infty} \int_{-\infty}^{\hat{x}} \int_{-\infty}^{\infty} \frac{\partial}{\partial z}(U_zU_x - A_zA_x)dz dx dy = 0, \end{aligned} \quad (\text{B.3})$$

where we have interchanged some of the integral signs. For the solutions of interest, U will be exponentially small and derivatives of A will be $O(r^{-2})$ when x, y, z are large positive or negative. Then, of the three triple integrals, it is clear that the last two are actually zero, as is evident when the inner integrals are evaluated. For the first triple integral, only the upper limit contributes when the inner integral is evaluated. Thus (B.3) becomes

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2}(U_x^2 - U_y^2 - U_z^2 - W(U^2) + (\omega - A)^2U^2 - A_x^2 + A_y^2 + A_z^2)dy dz = 0, \quad (\text{B.4})$$

where the integrand is to be evaluated at \hat{x} . For simplicity, however, we will regard \hat{x} to be rewritten as x . For present purposes we are primarily interested in the special case where (B.4) is also integrated over x so that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2}(U_x^2 - U_y^2 - U_z^2 - W(U^2) + (\omega - A)^2U^2 - A_x^2 + A_y^2 + A_z^2)dx dy dz = 0. \quad (\text{B.5})$$

When U and A are spherically symmetric, i.e., $U(x, y, z) = \mathcal{U}(r)$ and $A(x, y, z) = \mathcal{A}(r)$ with $r^2 = x^2 + y^2 + z^2$, (B.5) can be rewritten in terms of the integrals (8.6)–(8.9), and the result (10.2) follows. An advantage of the longer derivation in this appendix is that the more general result (B.4) is also obtained in addition to (10.2).

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